On triangles in K_r-minor free graphs [1]

Boris Albar [a], Daniel Gonçalves [b]

[a] I3M & LIRMM, CNRS and Univ. Montpellier 2, Place Eugène Bataillon, 34095 Montpellier Cedex 5, France

[b] LIRMM, CNRS and Univ. Montpellier 2, 161 rue Ada, 34095 Montpellier Cedex 5, France

Abstract

We study graphs where each edge adjacent to a vertex of small degree (7 and 9, respectively) belongs to many triangles (4 and 5, respectively) and show that these graphs contain a complete graph (K_6 and K_7, respectively) as a minor. The second case settles a problem of Nevo (Nevo, 2007). Morevover if each edge of a graph belongs to 6 triangles then the graph contains a K_8-minor or contains $K_{2,2,2,2,2}$ as an induced subgraph. We then show applications of these structural properties to stress freeness and coloration of graphs. In particular, motivated by Hadwiger's conjecture, we prove that every K_7-minor free graph is 8-colorable and every K_8-minor free graph is 10-colorable.

1 Introduction

A minor of a graph G is a graph obtained from G by a succession of edge deletions, edge contractions and vertex deletions. All graphs we consider are simple, i.e. without loops or multiple edges. The following theorem of Mader [13] bounds the number of edges in a K_r-minor free graph.

Theorem 1 (Mader, 1968, [13]) *For $3 \leq r \leq 7$, any K_r-minor free graph G on $n \geq r$ vertices has at most $(r-2)n - \binom{r-1}{2}$ edges.*

Note that since $|E(G)| = \frac{1}{2}\sum_{u \in V(G)} \deg(u)$, this theorem implies that every K_r-minor free graph G, for $3 \leq r \leq 7$, is such that $\delta(G) \leq 2r-5$, where $\delta(G)$

Email addresses: `boris.albar@lirmm.fr` (Boris Albar), `daniel.goncalves@lirmm.fr` (Daniel Gonçalves).

[1] This work was partially supported by the ANR grant EGOS 12 JS02 002 01

denotes the minimum degree of G. This property will be of importance in the following. We are interested in a sufficient condition for a graph to admit a complete graph as a minor, dealing with the minimum number of triangles each edge belongs to. Nevo [15] already studied this problem for small cliques. In the following, we assume that every graph has at least one edge.

Theorem 2 (Nevo, 2007, [15]) *For $3 \leq r \leq 5$, any K_r-minor free graph G has an edge that belongs to at most $r-3$ triangles.*

He also gave a weaker version for K_6-minor free graphs.

Theorem 3 (Nevo, 2007, [15]) *Any K_6-minor free graph G has an edge that belongs to at most $r-3$ triangles, or is a clique-sum over K_r, $r \leq 4$.*

Nevo has conjectured that Theorem 3 can be extended to the case of K_7-minor free graphs. We improve Theorems 2 and 3 in the following way.

Theorem 4 *For $3 \leq r \leq 7$, any K_r-minor free graph G has an edge uv such that $\deg(u) \leq 2r-5$ and uv belongs to at most $r-3$ triangles.*

In particular, this answers Nevo's conjecture about K_7-minor free graphs. As pointed out by Nevo, Theorem 3 cannot be further extended to K_8-minor free graphs as such, since $K_{2,2,2,2,2}$ is a K_8-minor free graph whose every edge belongs to 6 triangles. Actually, one can obtain K_8-minor free graphs whose every edge belongs to 6 triangles by gluing copies of $K_{2,2,2,2,2}$ on cliques of any K_8-minor free graph. It is interesting to notice that $K_{2,2,2,2,2}$ appears in a Mader-like theorem for K_8-minor free graphs [9].

Theorem 5 (Jørgensen, 1994, [9]) *Every graph on $n \geq 8$ vertices and at least $6n-20$ edges either has a K_8-minor, or is a $(K_{2,2,2,2,2}, 5)$-cockade (i.e. any graph obtained from copies of $K_{2,2,2,2,2}$ by 5-clique sums).*

Although Theorem 4 cannot be extended to K_8-minor free graphs, some similar conclusions can be reached by considering stronger hypotheses. By increasing the minimum degree of the graph or excluding $K_{2,2,2,2,2}$ as an induced subgraph, we have the following three theorems.

Theorem 6 *Any K_8-minor free graph G with $\delta(G) = 11$ has an edge uv such that u has degree 11 and uv belongs to at most 5 triangles.*

Theorem 7 *Any K_8-minor free graph G with $\delta(G) \geq 9$ has an edge that belongs to at most 5 triangles.*

Theorem 8 *Any K_8-minor free graph G with no $K_{2,2,2,2,2}$ as induced subgraph has an edge that belongs to at most 5 triangles.*

We investigate applications of the previous results in the rest of the paper. In Section 5, we relax the hypothesis into a more global condition on the overall number of triangles in the graph. In particular, we prove that, for $3 \leq k \leq 7$ (resp. $k = 8$), if a graph has $m \geq 1$ edges and at least $\frac{k-3}{2}m$ triangles, then it has a K_k-minor (resp. a K_8- or a $K_{2,2,2,2,2}$-minor). In Section 6, we show applications to stress freeness of graphs, and settle some open problems of Nevo [15]. Finally, we show some applications to graph coloration in Section 7 and Section 8. In the former section, we show an application to double-critical k-chromatic graphs which settle a special case of a conjecture of Kawarabayashi, Toft and Pedersen [10]. In the latter section, motivated by Hadwiger's conjecture, we show that every K_7-minor free graph is 8-colorable and that every K_8-minor free graph is 10-colorable.

2 Proof of Theorem 4 for $r \leq 6$: A slight improvement of Nevo's theorem

First note that the cases $r = 3$ or 4 are trivial. The case $r = 5$ is also quite immediate, but we need a few definitions to prove it. A *separation* of a graph G is a pair (A, B) of subsets of $V(G)$ such that $A \cup B = V(G)$, $A \setminus B \neq \emptyset$, $B \setminus A \neq \emptyset$, and no edge has one end in $A \backslash B$ and the other in $B \backslash A$. The *order* of a separation is $|A \cap B|$. A separation of order k will be denoted as a k-separation, and a separation of order at most k as a $(\leq k)$-separation. Given a vertex set $X \subseteq V(G)$ (eventually X is a singleton) the sets $N(X)$ and $N[X]$ are respectively defined by $\{y \in V(G) \setminus X \mid \exists x \in X \text{ s.t. } xy \in E(G)\}$ and $X \cup N(X)$.

Let us prove the case $r = 5$. Consider any K_5-minor free graph G. According to Wagner's characterization of K_5-minor free graphs [21], G is either the Wagner graph, a 4-connected planar graph, or has a (≤ 3)-separation (A, B) such that $H = G[A]$ is either the Wagner graph or a 4-connected planar graph. If G or H is the Wagner graph, as this graph has only degree 3 vertices and no triangle, we are done. If G (resp. H) is a 4-connected planar graph, Euler's formula implies that there is a vertex v of degree at most 5 in $V(G)$ (resp. in $A \setminus B$). One can then observe that, any edge around v belongs to at most 2 triangles, as otherwise there would be a separating triangle in G (resp. H), contradicting its 4-connectivity.

Let us now focus on the case $r = 6$ of Theorem 4. Consider by contradiction a K_6-minor free graph G with at least one edge, and such that every edge incident to a vertex of degree at most 7, belongs to at least 4 triangles. By Mader's theorem, we have that $\delta(G) \leq 7$. We start by studying the properties of $G[N(u)]$, for the vertices u of degree at most 7. First, it is clear that $G[N(u)]$ is K_5-minor free because otherwise there would be a K_6-minor in G,

contradicting the hypothesis.

Lemma 9 *$\delta(G) \geq 6$, and for any vertex u of degree at most 7, $\delta(G[N(u)]) \geq 4$.*

Proof. For any vertex u of degree at most 7, and any vertex of $v \in N(u)$ the edge uv belongs to at least 4 triangles. The third vertex of each triangle clearly belongs to $N(u)$ and is adjacent to v. Thus v has degree at least 4 in $G[N(u)]$.

Since for any vertex u of degree at most 7 we have $\delta(G[N(u)]) \geq 4$, $|N(u)| \geq 5$ (i.e. $\deg(u) \geq 5$). Furthermore if there was a vertex u of degree 5, as $\delta(G[N(u)]) \geq 4$, the graph $G[N(u)]$ would be isomorphic to K_5, contradicting the fact that $G[N(u)]$ is K_5-minor free. Thus $\delta(G) \geq 6$. □

As observed by Nevo (Proposition 3.3, [15]), since $|N(u)| \leq 7$, $\delta(G[N(u)]) \geq 4$ and $N(u)$ is K_5-minor free, then $G[N(u)]$ is 4-connected. Note that by Wagner's characterization of K_5-minor free graphs, every 4-connected K_5-minor free is planar. Chen and Kanevsky [3] proved that every 4-connected graph can be obtained from K_5 and the double-axle wheel W_4^2 by operations involving vertex splitting and edge addition. Their result implies that the only two possibilities for $G[N(u)]$ are the double-axle wheels on 4 and 5 vertices depicted in Figure 1. Note that theses two graphs have $3|N(u)|-6$ edges, and hence are maximal K_5-minor free (by Mader's theorem).

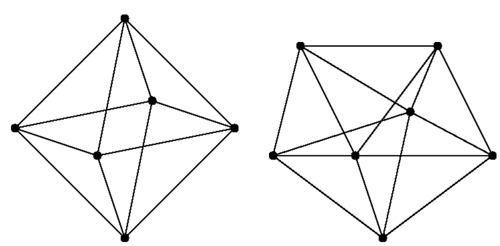

Fig. 1. The double-axle wheel on 4 and 5 vertices.

We need the following lemmas on the neighborhood of the vertices with small degree.

Lemma 10 *For any vertex u of degree at most 7, every vertex $v \in N(u)$ has a neighbor in $G \setminus N[u]$.*

Proof. Recall that $G[N(u)]$ is a double-axle wheel. Note that in a double-axle wheel, every vertex has degree at most 5, and every edge belongs to exactly 2 triangles. Thus, every vertex of $N(u)$ has degree at most 6 in $G[N[u]]$, and every edge of $G[N(u)]$ belongs to exactly 3 triangles in $G[N[u]]$. This implies that any vertex $v \in N(u)$ has either degree > 8 in G, and thus at least 2 neighbors in $G \setminus N[u]$, or that any of its incident edges vw in $G[N(u)]$ is

contained in a fourth triangle vwx, with $x \in G \setminus N[u]$. $\square$

Lemma 11 *For any vertex u of degree at most 7, and any connected component C of $G \setminus N[u]$, the graph $G[N(C)]$ is a clique on at most 3 vertices.*

Proof. As $G[N(u)]$ has no clique on more than 3 vertices, let us show that $N(C)$ does not contain two non-adjacent vertices , say v_1 and v_2. There exists a path from v_1 to v_2 with inner vertices in C. Since $G[N(u)]$ is maximal K_5-minor free, this path together with $G[N[u]]$ induces a K_6 minor in G, a contradiction. $\square$

Lemma 12 *For any vertex u of degree at most 7, and any connected component C of $G \setminus N[u]$, there exists a vertex $u' \in C$ of degree at most 7 in G.*

Proof. Suppose for contradiction that every vertex of C has degree at least 8 in G. Note that by definition, every vertex in $N(C)$ has a neighbor in C. Thus, as by Lemma 11 $G[N(C)]$ is a clique on $k \leq 3$ vertices, the vertices in $N(C)$ have degree at least k in $G[N[C]]$. Thus the number of edges of $G[N[C]]$ is at least

$$|E(G[N[C]])| \geq \frac{1}{2}(8|C| + k^2) > 4(|C| + k) - 10$$

and by Mader's theorem, there is a K_6-minor in $G[N[C]]$, a contradiction. $\square$

Now choose a vertex u of degree at most 7 and a connected component C of $G \setminus N[u]$, in such a way that $|C|$ is minimum. By Lemma 12, C has a vertex v of degree at most 7.

Let C_u be the connected component of $G \setminus N[v]$ that contains u, and let $x \in N(v) \setminus N(C_u)$. By Lemma 10, there is a connected component C' of $G \setminus N[v]$ such that $x \in N(C')$.

As $N[u] \subset N[C_u]$, it is clear that $G[C' \cup \{x, v\}]$ is a connected subgraph of $G \setminus N[u]$. We thus have that $C' \subsetneq C$ and thus that $|C'| < |C|$, contradicting the choice of u and C. This concludes the proof of the case $r = 6$ of Theorem 4.

3 Proof of Theorem 4 for $r = 7$: the case of 5 triangles

Consider by contradiction a K_7-minor free graph G with at least on edge, and such that every edge incident to a vertex of degree at most 9 belongs to at

least 5 triangles. By Mader's theorem, $|E(G)| \leq 5|V(G)| - 15$, hence there are vertices u such that $\deg(u) \leq 9$.

We start by studying the properties of $G[N(u)]$, for any vertex u of degree at most 9. First, it is clear that $G[N(u)]$ is K_6-minor free because otherwise there would be a K_7-minor in G, contradicting the hypothesis.

Lemma 13 *$\delta(G) \geq 7$, and for any vertex u of degree at most 9, $\delta(G[N(u)]) \geq 5$.*

Proof. For any vertex u of degree at most 9, and any vertex of $v \in N(u)$ the edge uv belongs to at least 5 triangles. The third vertex of each triangle clearly belongs to $N(u)$ and is adjacent to v. Thus v has degree at least 5 in $G[N(u)]$.

Since for any vertex u of degree at most 9 we have $\delta(G[N(u)]) \geq 5$, $|N(u)| \geq 6$ (i.e. $\deg(u) \geq 6$). Furthermore if there was a vertex u of degree 6, as $\delta(G[N(u)]) \geq 5$, the graph $G[N(u)]$ would be isomorphic to K_6, contradicting the fact that $G[N(u)]$ is K_6-minor free. Thus $\delta(G) \geq 7$. $\square$

There is no appropriate theorem (contrarily to the previous case) to generate all possible neighbourhoods of the small degree vertices. Instead, we use a computer to generate all graphs with at most 9 vertices and minimum degree at least 5. Then we refine (by computer) our list of graphs, by removing the ones having a K_6-minor. At the end, we end up with a list of 22 graphs. A difference with the previous case is that not all the 22 graphs are maximal K_6-minor free graphs. We deduce two of the following lemmas from the study of $N(u)$ by computer [1].

Lemma 14 *For any vertex u of degree at most 9, any connected component C of $G \setminus N[u]$ is such that $|N(C)| = k \leq 5$ and $|E(N(C))| \geq \binom{k}{2} - 3$ (i.e. $G[N[C]]$ has at most 3 non-edges).*

Proof. As any connected component C could be contracted into a single vertex, we prove the lemma by attaching a new vertex to all possible combinations of k vertices of $N[u]$ (as we know that $N(u)$ induces one of the 22 graphs generated above), for any $k \leq 6$, and check when it induces a K_7-minor. $\square$

This allows us to prove the following equivalent of Lemma 12.

Lemma 15 *For any vertex u of degree at most 9, any connected component C of $G \setminus N[u]$ has a vertex u' of degree at most 9 in G.*

Proof. Let u be a vertex of G of degree at most 9 and let C be a connected

component of $G \setminus N[u]$ which vertices have degree at least 10 in G. Note that by definition every vertex of $N(C)$ has at least one neighbor in C. Lemma 14 implies that $|N(C)| = k \leq 5$ and that $G[N(C)]$ has at most 3 non-edges. Thus, contracting a conveniently choosen edge between u and $N(C)$, one obtains that $G[N(C)]$ has at most 1 non-edge. After this contraction, we have:

$$\begin{aligned} |E(N[C])| &\geq \frac{1}{2}\Big[10|C| + k(k-1) - 2 + k\Big] \\ &= 5|C| + \frac{k^2}{2} - 1 > 5(|C| + k) - 15. \end{aligned}$$

This contradicts the fact that $G[N[C]]$ is K_7-minor free, and thus concludes the proof of the lemma. □

Lemma 16 *For any vertex u of degree at most 9, at most one vertex v of $N(u)$ is such that $N(v) \subseteq N[u]$.*

Proof. For every such vertex v, as $\deg(v) \leq \deg(u) \leq 9$, the edges adjacent to v with both ends in $N(u)$ belong to at least 5 triangles in G (i.e. belong to at least 4 triangles in $G[N(u)]$). We checked that for every graph in the list at most one such vertex satisfies this condition. □

This allows us to prove the following lemma.

Lemma 17 *For any vertex u of degree at most 9 and any connected component C of $G \setminus N[u]$, there exists a connected component C' of $G \setminus N[u]$ such that $N(C') \setminus N(C) \neq \emptyset$.*

Proof. As $\deg(u) \geq 7$ (by Lemma 13) and $|N(C)| \leq 5$ (by Lemma 14), there are at least 2 vertices in $N(u) \setminus N(C)$. By Lemma 16, one of these 2 vertices has a neighbor x out of $N[u]$. Thus the component of $G \setminus N[u]$ containing x fulfills the requirements of the lemma. □

Now choose a vertex u of degree at most 9 and a connected component C of $G \setminus N[u]$, in such a way that $|C|$ is minimum. By Lemma 15, C has a vertex v of degree at most 9. Let C_u be the connected component of $G \setminus N[v]$ that contains u. By Lemma 17 there exists a connected component C' of $G \setminus N[v]$ such that $N(C') \setminus N(C_u) \neq \emptyset$, and let $x \in N(C') \setminus N(C_u)$. As $N[u] \subset N[C_u]$, it is clear that $G[C' \cup \{x, v\}]$ is a connected subgraph of $G \setminus N[u]$. We thus have that $C' \subsetneq C$ and thus that $|C'| < |C|$, contradicting the choice of u and C. This concludes the proof of case $r = 7$ of Theorem 4

4 Proof of Theorem 6, 7 and 8 : the case of 6 triangles

As in the previous sections, we will consider vertices of small degree (and their neighborhoods) in K_8-minor free graphs. We thus need the following technical lemma that has been proven by computer [1].

Lemma 18 *Every K_7-minors free graph H distinct from $K_{2,2,2,2}$, $K_{3,3,3}$ and $\overline{P_{10}}$ (the complement of the Petersen graph), and such that $8 \leq |V(H)| \leq 11$ and $\delta(H) \geq 6$, verifies:*

- *H is 5-connected.*
- *H has at most one vertex v such that each of its incident edges belongs to 5 triangles.*
- *For any subset $Y \subsetneq V(H)$ of size 7, the graph obtained from H by adding two vertices x and y such that $N(x) = V(H)$ and $N(y) = Y$, has a K_8-minor.*

Note that the second property also holds for $K_{2,2,2,2}$, $K_{3,3,3}$ and $\overline{P_{10}}$. Actually any edge of these 3 graphs belongs to less than 5 triangles.

By Theorem 5, any K_8-minor free graph has minimum degree at most 11. Theorem 6 considers the case where the minimum degree is exactly 11. It will be used in Section 8 to color K_8-minor free graphs.

Proof of Theorem 6. We prove this using the same technique as in Section 3. Consider by contradiction a K_8-minor free graph G with $\delta(G) = 11$, and such that every edge adjacent to a degree 11 vertex belongs to at least 6 triangles. We start by studying the properties of $G[N(u)]$, for any degree 11 vertex u. First, it is clear that $G[N(u)]$ is K_7-minor free because otherwise there would be a K_8-minor in G, contradicting the hypothesis.

Lemma 19 *For any degree 11 vertex u, $\delta(G[N(u)]) \geq 6$.*

Proof. For any degree 11 vertex u and any vertex of $v \in N(u)$, the edge uv belongs to at least 6 triangles. The third vertex of each triangle clearly belongs to $N(u)$ and is adjacent to v. Thus v has degree at least 6 in $G[N(u)]$. □

Lemma 20 *For any degree 11 vertex u, any connected component C of $G \setminus N[u]$ has a vertex u' of degree at most 11 in G.*

Proof. Let u be a degree 11 vertex of G and let C be any connected component of $G \setminus N[u]$ which vertices have degree at least 12 in G. Lemma 18 implies that $G[N(u)]$ is 5-connected and that $|N(C)| = k \leq 6$. Thus the lemma holds by considering the graph $G[N[u] \cup C]$ in the following Lemma 21. □

Lemma 21 *A graph H with a degree 11 vertex $u \in V(H)$ and such that:*

(A) $H[N(u)]$ is 5-connected,
(B) $\delta(H[N(u)]) \geq 6$,
(C) the set $C = V(H) \setminus N[u]$ is non-empty, and all its vertices have degree at least 12, and
(D) the set $N(C) \subseteq N(u)$ has size $k \leq 6$,

has a K_8-minor.

Proof. Consider a minimal counter-example H, that is a K_8-minor free graph H fulfilling conditions (A), (B) (C) and (D), and minimizing $|V(H)|$. Note that by definition every vertex of $N(C) \subseteq N(u)$ has at least one neighbor in C. Let us prove that actually every vertex of $N(C)$ has at least 2 neighbors in C. If $x \in N(C)$ has only one neighbor y in C, contract the edge xy and denote G' the obtained graph. It is clear that G' is K_8-minor free, and fulfills conditions (A), (B) and (D). Moreover, $C \setminus \{y\}$ is non-empty as it contains at least 6 vertices of $N(y) \cap C$ (as $\deg(y) \geq 12$ and $|N(C)| = k \leq 6$), and every vertex of $C \setminus \{y\}$ has degree at least 12 in H' as none of these vertices are adjacent to x in H. So G' also fulfills condition (C), and this contradicts the minimality of G. Thus every vertex of $N(C)$ has at least 2 neighbors in C.

One can easily see that every $(K_{2,2,2,2,2}, 5)$-cockade has at least 10 degree 8 vertices. Thus the graph $H[N[C]]$, and any graph obtained from $H[N[C]]$ by adding edges, cannot be a $(K_{2,2,2,2,2}, 5)$-cockade as it has at most 6 vertices of degree 8. Thus as $H[N[C]]$ has at least $\frac{1}{2}(12|C| + 2k)$ edges and as this is at least $6(|C| + k) - 20$ for $k \leq 4$, by Theorem 5 we have that $5 \leq k \leq 6$.

Now suppose that $k = 5, 6$. Let v_1 and v_2 be two vertices of smallest degree in $H[N(C)]$. Denote δ_1 and δ_2 their respective degree in $H[N(C)]$. Note that if $k = 6$ then $\delta_1 \geq 1$ as v_1 has at least 6 neighbors in $N(u)$ and as there are only 5 vertices in $N(u) \setminus N(C)$. By contracting the edge uv_1, we have $k - 1 - \delta_1$ additionnal edges in $H[N[C]]$. Moreover since $H[N(u)]$ is 5-connected and since $|N(C)| \leq 6$, for every vertex $x \neq v_2$ of $N(C)$ we have $|N(C) \setminus \{x, v_2\}| = 4$ and thus the graph $H[N(u)] \setminus (N(C) \setminus \{x, v_2\})$ is connected. Thus, iteratively contracting all the edges between v_2 and $N(u) \setminus N(C)$ we add at least $k-2-\delta_2$ edges in $H[N[C]]$ (as we have potentially already added the edge v_1v_2 in the previous step). The number of edges in the obtained graph is at least

$$\frac{1}{2}[(\delta_1 + 2) + (\delta_2 + 2)(k - 1)) + 12|C|] + (k - 1 - \delta_1) + (k - 2 - \delta_2)$$

which is more than $6(|C| + k) - 20$ (as $k \leq 6$ and as if $k = 6$ then $\delta_1 \geq 1$). Thus this graph has a K_8-minor, and so does H. This completes the proof of the lemma. $\square$

Lemma 22 *For any degree 11 vertex u and any connected component C of $G \setminus N[u]$, there exists a connected component C' of $G \setminus N[u]$ such that $N(C') \setminus N(C) \neq \emptyset$.*

Proof. As $\deg(u) = 11$ and $|N(C)| \leq 6$ (by Lemma 18), there are at least 5 vertices in $N(u) \setminus N(C)$. As $\delta(G) = 11$ one can easily derive from Lemma 18 that one (actually, at least 4) of these vertices has a neighbor x out of $N[u]$. Thus the component of $G \setminus N[u]$ containing x fulfills the requirements of the lemma. □

Now choose a degree 11 vertex u and a connected component C of $G \setminus N[u]$, in such a way that $|C|$ is minimum. By Lemma 20, C has a degree 11 vertex v. Let C_u be the connected component of $G \setminus N[v]$ that contains u. By Lemma 22 there exists a connected component C' of $G \setminus N[v]$ such that $N(C') \setminus N(C_u) \neq \emptyset$, and let $x \in N(C') \setminus N(C_u)$. As $N[u] \subset N[C_u]$, it is clear that $G[C' \cup \{x, v\}]$ is a connected subgraph of $G \setminus N[u]$. We thus have that $C' \subsetneq C$ and thus that $|C'| < |C|$, contradicting the choice of u and C. This concludes the proof of Theorem 6 □

let us now prove Theorem 8. Given a counter-exemple G of Theorem 8, note that adding a vertex s to G, adjacent to a single vertex of G, one obtains a counter-exemple of the following theorem, thus Theorem 8 is a corollary of the following theorem.

Theorem 23 *Consider a connected K_8-minor free graph G with a vertex s of degree at most 7 and such that $N[s] \subsetneq V(G)$. If every edge $e \in E(G) \setminus E(G[N[s]])$ belongs to at least 6 triangles, then G contains an induced $K_{2,2,2,2,2}$.*

Note that as $K_{2,2,2,2,2}$ is maximal K_8-minor free, any K_8-minor free graph G containing a copy of $K_{2,2,2,2,2} = G[X]$, for some vertex set $X \subseteq V(G)$, is such that any connected component C of $G \setminus X$ verifies that $N(C)$ induces a clique in $G[X]$.

Proof. Consider a connected K_8-minor free graph G with a vertex s of degree at most 7 such that $N[s] \subsetneq V(G)$, such that G does not contain an induced $K_{2,2,2,2,2}$, and such that every edge $e \in E(G) \setminus E(G[N[s]])$ belongs to at least 6 triangles. Assume also that G minimizes the number of vertices. This property implies that $G \setminus N[s]$ is connected. Indeed, otherwise one could delete one of the connected components in $G \setminus N[s]$ and obtain a smaller counter-example. The graph G is almost 8-connected as observed in the following lemma.

Lemma 24 *For any separation (A, B) of G (denote $S = A \cap B$), we have either:*

- $|S| \geq 8$, *or*
- $s \notin S$ *and* $A \setminus B = \{s\}$ *(i.e.* $B = V(G) \setminus \{s\}$*), or*
- $s \in S$ *and* $|S| \geq 6$.

Proof. Suppose there exists a separation (A, B) contradicting the lemma. Note that $|S| < 8$ and let us assume that $s \in A$.

Consider first the case where $s \notin S = A \cap B$, that is the case where $\{s\} \subsetneq A \setminus B$. Assume that among all such counter-examples, (A, B) minimizes $|S|$. In this case, if the connected component of $A \setminus B$ containing s has more vertices then, contracting this component into s, one obtains a proper minor G' of G such that $N[s] \subsetneq V(G')$ (as $B \setminus A \neq \emptyset$) and such that every edge not in $E(N[s])$ belongs to 6 triangles. This would contradict the minimality of G, and we thus assume the existence of a component $C_0 = \{s\}$ in $G \setminus B$. As $\{s\} \subsetneq A \setminus B$, let $C_1 \neq \{s\}$ be some connected component of $G \setminus B$. Let also C_2 be some component of $G \setminus A$. Note that for any of these components C_i, $N(C) \subsetneq S$. Otherwise one could contract (if needed) all the component into a single vertex s' and the graph induced by $\{s\} \cup N[C_1]$ or by $\{s\} \cup N[C_2]$ (a proper minor of G) would be a smaller counter-example. Note now that $S = N(s) \cup N(C_i)$ for $i = 1$ or 2. Indeed, otherwise the separation $(N[s] \cup N[C_i], V(G) \setminus (C_i \cup \{s\}))$ would be a counter-example contradicting the minimality of S. Finally note that $N(C_2) \not\subseteq N(C_1)$, as otherwise contracting C_1 into a single vertex s' and considering the graph induced by $C_2 \cup \{s'\}$ one would obtain a smaller counter-example. Thus there exists a vertex $x \in N(s) \cap N(C_2)$ such that $x \notin N(C_1)$. Contracting the edge xs and contracting the whole component C_2 into x, and considering the graph induced by $C_1 \cup \{x\}$ one obtains a smaller counter-example (where x plays the role of s).

Consider now the case where $s \in S = A \cap B$ and note that $|S| < 6$. Assume that among all the separations containing s, (A, B) minimizes $|S|$. Note that every connected component C of $G \setminus S$ is such that $s \in N(C)$. Indeed, we have seen above that otherwise C would be such that $|N(C)| \geq 8$ (by considering the separation $(V(G) \setminus C, N[C])$ and noting that $\{s\} \subsetneq V(G) \setminus N[C]$), and this would contradict the fact that $|S| < 6$. This implies that every connected component C of $G \setminus S$ is such that $N(C) = S$. Otherwise $(V(G) \setminus C, N[C])$ would be a separation containing s contradicting the minimality of S. We can assume without loss of generality that s has at most as many neighbors in $B \setminus A$ than in $A \setminus B$. In particular, since $\deg(s) \leq 7$, s has at most 3 neighbors in $B \setminus A$. Note that $B \not\subseteq N[s]$ as otherwise $G \setminus (B \setminus A)$ would be a smaller counter-example. Thus there is an edge in $G[B \setminus N[s]$ that belongs to at least 6 triangles, and thus $|B| \geq 9$ (s and the 6 triangles). Thus contracting every component of $A \setminus B$ on s, results in a proper minor G' of G such that $\deg(s) \leq 7$ (at most 4 in S and 3 in $B \setminus A$), such that $N[s] \subsetneq V(G')$ (as $|V(G')| = |B| \geq 9$), and such that every edge not in $E(N[s])$ belongs to 6 triangles, contradicting the minimality of G. This concludes the proof of the

lemma. $\square$

By Theorem 5 G has at most $6n-20$ edges, and thus there are several vertices in G with degree at most 11. Let us prove that there are such vertices out of $N[s]$.

Lemma 25 *There are at least 2 vertices in $V(G) \setminus N[s]$ with degree at most 11.*

Proof. Assume for contradiction that every vertex of $V(G) \setminus N[s]$ but one, say x, has degree at least 12, and recall that such vertex has degree at least 8. Note that every vertex $v \in N(s)$ has a neighbor in $V(G) \setminus N[s]$, as otherwise $G \setminus v$ would be a smaller counter-exemple. Thus every vertex $v \in N(s)$ has an incident edge that belongs to at least 6 triangles (without using the edge sv), which implies that $\deg(v) \geq 8$. This implies that the number of edges in G verifies :

$$12n - 42 \geq 2|E(G)| = \sum_{v \in V(G)} \deg(v) \geq 8 + k + 8k + 12(n - k - 2)$$

where $k = \deg(s)$. This implies that $3k \geq 26$ which contradicts the fact that $k = \deg(s) \leq 7$. This concludes the proof of the lemma. $\square$

As for any vertex $u \in V(G) \setminus N[s]$ each of its incident edges belongs to 6 triangles, the graph $G[N(u)]$ has minimum degree at least 6. As G does not contain K_8 as subgraph, this also implies that $\deg(u) \geq 8$. So there are at least two vertices in $V(G) \setminus N[s]$ with degree between 8 and 11. The next lemma tells us more on the neighborhood of these small degree vertices.

Lemma 26 *For every vertex $u \in V(G) \setminus N[s]$ with degree at most 11 in G, $G[N(u)]$ is isomorphic to $K_{2,2,2,2}$, $K_{3,3,3}$ or $\overline{P_{10}}$.*

Proof. Let u be any vertex of $V(G) \setminus N[s]$ with degree at most 11 in G. As observed earlier $8 \leq \deg(u) \leq 11$ and $\delta(G[N(u)]) \geq 6$. Assume for contradiction that $N(u)$, is not isomorphic to $K_{2,2,2,2}$, $K_{3,3,3}$ or $\overline{P_{10}}$. Note that $|N(u) \cap N(s)| \leq 6$, as otherwise Lemma 18 would contradict the K_8-minor freeness of G.

By Lemma 18 one of the (at least two) vertices in $N(u) \setminus N(s)$, say x, has an incident edge in $G[N(u)]$ that belongs to at most 5 triangles in $G[N[u]]$. Thus the sixth triangle containing this edge goes through a vertex v of $V(G) \setminus (N[u] \cup \{s\})$.

Lemma 24 implies that the connected component C of v in $V(G) \setminus N[u]$ is such that $N(C) \geq 8$. The graph obtained by contracting C into a single vertex has

a K_8-minor (by Lemma 18), a contradiction. $\square$

A K_3-minor rooted at $\{a, b, c\}$, or a $\{a, b, c\}$-minor, is a K_3-minor in which you can contract edges incident to a, b or c, to obtain a K_3 with vertex set $\{a, b, c\}$. For the rest of the proof we need the following characterization of rooted K_3-minor.

Theorem 27 (D. R. Wood and S. Linusson, Lemma 5 of [24]) *For distinct vertices a, b, c in a graph G, either:*

- *G contains an $\{a, b, c\}$-minor, or*
- *for some vertex $v \in V(G)$ at most one of a, b, c are in each component of $G \setminus v$.*

Lemma 28 *For every vertex $u \in V(G) \setminus N[s]$ with degree at most 11 in G, the graph $G[N(u)]$ is not isomorphic to $K_{3,3,3}$.*

Proof. Observe that adding two vertex disjoint edges or three edges of a triangle in $K_{3,3,3}$ yields a K_7-minor. Now assume for contradiction that there exists some vertex $u \in V(G) \setminus N[s]$ such that $G[N(u)]$ is isomorphic to $K_{3,3,3}$.

As the set $N(u) \setminus N[s]$ is non-empty (it has size at least $9 - 7$) and as every vertex v in $N(u) \setminus N[s]$ has degree at least 8, and thus has a neighbor out of $N[u]$, $G \setminus N[u]$ has a connected component $C \neq \{s\}$. By Lemma 24 $|N(C)| \geq 8$.

If $G \setminus N[u]$ has another connected component C' such that $|N(C')| \geq 6$, one can create two vertex disjoint edges in $K_{3,3,3}$ by contracting two vertex disjoint paths with non-adjacent ends in $N(u)$, one living in each component. This would contradict the K_8-minor freeness of G. Thus if there is a component C', we should have $C' = \{s\}$ and $\deg(s) \leq 5$, as by Lemma 24 a component $C' \neq \{s\}$ would be such that $|N(C')| \geq 8$. In the following we consider the graph $G' = G[N[u] \cup C]$ (which is G or $G \setminus s$).

Let $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$ and $\{c_1, c_2, c_3\}$ be the three disjoint stables of $N(u) = K_{3,3,3}$. Without loss of generality we can assume that $\{a_1, a_2, a_3\} \subset N(C)$, and that $a_1 \notin N(s)$. As the edges of $N(u)$ incident to a_1 belong to at least 6 triangles, a_1 has at least two neighbors in $G' \setminus N[u]$. By Theorem 27 (applied to $\{a_1, a_2, a_3\}$ in the graph $G'' = G' \setminus \{u, b_1, b_2, b_3, c_1, c_2, c_3\}$), there is a vertex $v \in V(G'')$ such that at most one of a_1, a_2, a_3 are in each component of $G'' \setminus v$. Note that since a_1, a_2 and $a_3 \in N(C)$, all the sets $C \cup \{a_i, a_j\}$ induce a connected graph, and thus $v \neq a_1$, a_2 or a_3. Equivalently we have that $v \in V(G') \setminus N[u]$. Hence $G'' \setminus \{v\}$ contains at least 3 components C_1, C_2 and C_3 with $a_i \in C_i$, for $1 \leq i \leq 3$. Since a_1 has at least two neighbors in $G' \setminus N[u]$, one of them is distinct from v and we can define C'_1 as a connected component of $C_1 \setminus \{a_1\}$. Note that by construction $N(C'_1) \subset N(u) \cup \{v\}$. Since $C'_1 \neq \{s\}$

(as $a_1 \notin N(s)$) and as we might have $v = s$, Lemma 24 implies that $N(C_1') \geq 6$ (including v and a_1). So C_1' has at least 4 neighbors in $\{b_1, b_2, b_3, c_1, c_2, c_3\}$ and there is a path with interior vertices in C_1' between two vertices b_i and b_j, or between two vertices c_i and c_j. Furthermore, there is a path with interior vertices in $C_2 \cup \{v\} \cup C_3$ between the vertices a_2 and a_3. This contradicts the K_8-minor freeness of G, and thus concludes the proof of the lemma. □

Lemma 29 *For every vertex $u \in V(G) \setminus N[s]$ with degree at most 11 in G, the graph $G[N(u)]$ is not isomorphic to $K_{2,2,2,2}$.*

Proof. Assume for contradiction that there exists some vertex $u \in V(G) \setminus N[s]$ such that $G[N(u)]$ is isomorphic to $K_{2,2,2,2}$. One can check that adding two edges in $K_{2,2,2,2}$ creates a K_7-minor. Thus as G is K_8-minor free it should not be possible to add (by edge contractions) two new edges in $N(u)$.

Claim 30 *A vertex $v \in V(G) \setminus N[u]$ has at most six neighbors in $N(u)$.*

Proof. If there was a vertex v with 8 neighbors in $N(u)$, $N[u] \cup \{v\}$ would induce a $K_{2,2,2,2,2}$, a contradiction to the definition of G. We thus assume for contradiction that there is a vertex v with exactly 7 neighbors in $N(u)$. Note that eventually $v = s$. Let us denote x the only vertex in $N(u) \setminus N(v)$. Note that among the 4 non-edges of $G[N(u)]$, only one cannot be created by contracting an edge incident to v. So if there is a path whose ends are non-adjacent in $N(u)$ and whose inner vertices belong to $V(G) \setminus (N[u] \cup \{v\})$, then we have a K_8-minor, a contradiction. There is clearly such path if $s \neq v$ and if s has 5 neighbors in $N(u)$, we thus have that either $s = v$ or s has at most 4 neighbors in $N(u)$. Both cases imply that some edge xy (incident to x) does not belong to $G[N[s]]$, and thus xy belongs to at least 6 triangles. As xy belongs to only 5 triangles in $G[N[u]]$, this implies the existence of a vertex $w \in V(G) \setminus N[u]$ adjacent to x such that $w \neq s, v$. Let C be the connected component of w in $G \setminus (N[u] \cup \{v\})$. As $C \neq \{s\}$, Lemma 24 implies that $N(C)$ has size at least 6. Thus C has at least 5 neighbors in $N(u)$ and one can link two non-adjacent vertices of $N(u)$ by a path going through C, a contradiction. □

By Lemma 25 there exists another vertex $u' \in V(G) \setminus N[s]$ such that $\deg(u') \leq$ 11. By Lemma 26 and Lemma 28, $G[N(u')]$ is isomorphic to $K_{2,2,2,2}$ or $\overline{P_{10}}$.

Claim 31 *The vertices u and u' are non-adjacent.*

Proof. We assume for contradiction that u and u' are adjacent and we first consider the case where $G[N(u')]$ is isomorphic to $K_{2,2,2,2}$. In this case, as u' has allready 7 neighbors in $N[u]$, u' has a exactly one neighbor v in $G \setminus N[u]$. As v has 7 neighbors in $N(u')$, we have that $|N(u) \cap N(v)| \geq 7$, a contradiction to Claim 30.

If $G[N(u')]$ is isomorphic to $\overline{P_{10}}$, this implies that $G[N(u) \cap N(u')]$ is isomorphic to $\overline{C_6}$ (the complement of the 6-cycle). This is not compatible with $G[N(u)]$ being isomorphic to $K_{2,2,2,2}$, as this in turn implies that $G[N(u) \cap N(v)]$ is isomorphic to $K_{2,2,2}$. □

As by Lemma 24 there is no ($\leq$ 5)-separator (A, B) with $u \in A \setminus B$ and $u' \in B \setminus A$, Menger's Theorem implies the existence of 6 vertex disjoint paths between u and u'. These paths induces 6 disjoint paths $P_1 \dots P_6$ between $N(u)$ and $N(u')$. Note that every vertex in $N(u) \cap N(u')$ can be seen as a path of length 0.

Therefore, since $N(u)$ is isomorphic to $K_{2,2,2,2}$, there are two non-edges a_1a_2 and a_3a_4 of $G[N(u)]$ such that each a_i is the end of the path P_i. We denote by b_i, $1 \leq i \leq 4$ the end in $N(u')$ of the path P_i. Note that if $a_i \in N(u) \cap N(u')$ then $a_i = b_i$. Moreover we can suppose that the choice of a_1a_2 and a_3a_4 maximizes the size of $\{a_1, a_2, a_3, a_4\} \cap N(u')$. Since $N(u)$ is isomorphic to $K_{2,2,2,2}$ and since $|N(u) \cap N(u')| \leq 6$ (by Claim 30), there are at most two vertices in $N(u) \cap N(u')$ distinct from a_1, a_2, a_3, and a_4. Let $X = (N(u) \cap N(u')) \setminus \{a_1, a_2, a_3, a_4\}$.

Since both $K_{2,2,2,2}$ and $\overline{P_{10}}$ are 6-connected then $N[u']$ is 7-connected and so $G[N[u'] \setminus X]$ is 5-connected. Moreover $G[N[u'] \setminus X]$ has too many edges to be planar. Indeed, it has $9 - |X|$ vertices and at least $32 - 7|X|$ edges, which is more than $3(9 - |X|) - 6$ for $0 \leq |X| \leq 2$. We now need the following theorem of Robertson and Seymour about vertex disjoint pairs of paths.

Theorem 32 (Robertson and Seymour [18]) *Let $v_1, \dots, v_k$ be distinct vertices of a graph H. Then either*

(i) there are disjoint paths of H with ends p_1 p_2 and q_1 q_2 respectively, so that p_1, q_1, p_2, q_2 occur in the sequence $v_1, \dots, v_k$ in order, or
(ii) there is a (≤ 3)-separation (A, B) of H with $v_1, \dots, v_k \in A$ and $|B \setminus A| \geq 2$, or
(iii) H can be drawn in a disc with $v_1, \dots, v_k$ on the boundary in order.

Applying this theorem to the graph $G[N[u'] \setminus X]$ with $(v_1, \dots v_k) = (b_1, b_3, b_2, b_4)$ one obtains that there are two vertex disjoint paths in $N[u'] \setminus X$, a path $P_{1,2}$ between b_1 and b_2, and a path $P_{3,4}$ between b_3 and b_4. Theses paths are disjoint from $N[u]$ by construction, except possibly at their ends. Finally, since the paths P_i, for $1 \leq i \leq 4$, constructed above are disjoint from $N[u]$ and from $N[u'] \setminus X$, except at their ends, there exists two disjoint paths respectively linking a_1 with a_2 (through P_1, $P_{1,2}$ and P_2), and a_3 with a_4 (through P_3, $P_{3,4}$ and P_4). This contradicts the K_8-minor freeness of G and thus concludes the proof of the lemma. □

By Lemma 25 there exists at least two vertices u and $u' \in V(G) \setminus N[s]$ with degree at most 11. By Lemma 26, Lemma 28, and Lemma 29, both $G[N(u)]$ and $G[N(u')]$ are isomorphic to $\overline{P_{10}}$. The two graphs induced by $N[u]$ and $N[u']$ are close to a K_8-minor as observed in the following claim.

Claim 33 *In $\overline{P_{10}}$, adding two edges ab cd, such that ab, bc and $cd \notin E(\overline{P_{10}})$, creates a K_7-minor. Furthermore adding three edges e_1 e_2 and e_3, such that $e_1 \cap e_2 \cap e_3 = \emptyset$ in $\overline{P_{10}}$, creates a K_7-minor.*

Proof. One can easily check the accuracy of the first statement, by noting that adding any such pair of edges ab and cd, yields the same graph, and by noting that adding the edges u_1u_2 and u_3u_4 in $\overline{P_{10}}$ (notations come from Figure 2) the partition $\{\{0,2\},\{1\},\{3\},\{4\},\{5\},\{6,7\},\{8,9\}\}$ induces a K_7-minor.

For the second statement, we can assume that the three added edges are such that they pairwise do not correspond to the first statement. Without loss of generality, assume that one of the three edges is u_0u_5, and note that the other added edges are distinct from u_1u_2, u_1u_6, u_3u_4, u_4u_9, u_2u_7, u_7u_9, u_3u_8 and u_6u_8. Consider the case where one of the other added edges is incident to u_0u_5. By symmetry one can assume that this edge is u_0u_1, but this implies that the third added edge is distinct from u_0u_4 (as the three edges would intersect), and from u_3u_4, u_6u_9, u_5u_7 and u_5u_8 (by the first statement). There is thus no remaining candidate for the third edge. This implies that it is sufficient to consider the case where the edges u_0u_5, u_2u_3 and u_6u_9 are added in $\overline{P_{10}}$. In this case the partition $\{\{1\},\{4\},\{7\},\{8\},\{0,5\},\{6,9\},\{2,3\}\}$ induces a K_7-minor. $\square$

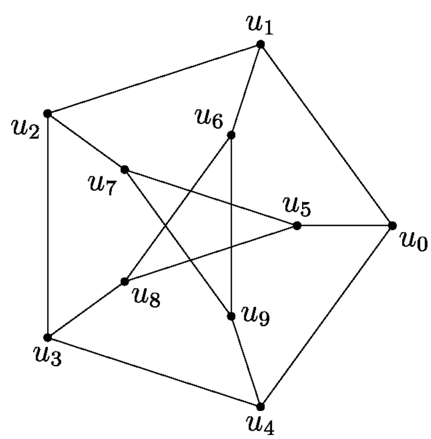

Fig. 2. The Petersen graph P_{10}.

Let us list the induced subgraphs of $\overline{P_{10}}$ of size 6.

Claim 34 *There are exactly 6 distinct induced subgraphs of size 6 in $\overline{P_{10}}$, including $K_{2,2,2}$. The complements of these graphs are represented in Figure 3. Furthermore note that every induced subgraphs of $\overline{P_{10}}$ of size at least 7, has a*

subgraph of size 6 distinct from $K_{2,2,2}$.

We do not prove the claim here as one can easily check its accuracy.

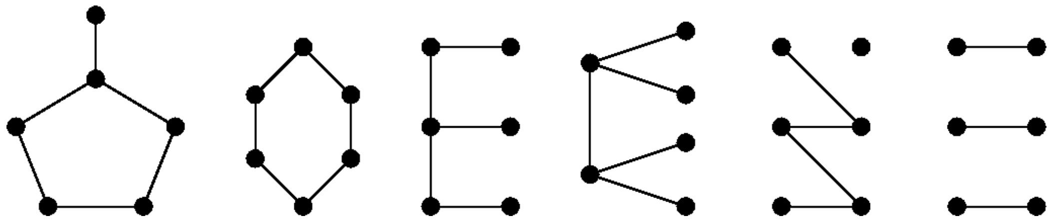

Fig. 3. The complements of the subgraphs of $\overline{P_{10}}$ of size 6 (i.e. the subgraphs of P_{10} of size 6).

Lemma 35 *The vertices of $N(u) \setminus N(s)$ (resp. of $N(u') \setminus N(s)$) have degree at least 12. Thus in particular, u and u' are non-adjacent.*

Proof. We assume for contradiction that u has a neighbor v of degree at most 11. By Lemma 26, Lemma 29, and Lemma 28, the graph $G[N(v)]$ is isomorphic to $\overline{P_{10}}$.

Assume $v = u_0$ in Figure 2. Since $N(u_0) \supset \{u, u_2, u_3, u_6, u_7, u_8, u_9\}$, the adjacencies in $G[\{u, u_2, u_3, u_6, u_7, u_8, u_9, \}]$ allow us to denote u by u'_0, and denote u'_1, u'_4 and u'_5 the vertices in $N(u_0) \setminus N[u]$, in such a way that these indices again correspond to Figure 2. It is now easy to see that contracting one edge in each of the paths (u_2, u'_4, u_7) and (u_6, u'_5, u_9) creates the edges u_2u_7 and u_6u_9 in $G[N[u]]$ and thus yields a K_8-minor (by Claim 33 as u_7u_9 is a non-edge of $\overline{P_{10}}$), a contradiction. □

The vertices u and u' are non-adjacent, however they can share neighbors. Let us prove that they cannot share more than 7 neighbors.

Lemma 36 $|N(u) \cap N(u')| \leq 7$.

Proof. Assume for contradiction that $|N(u) \cap N(u')| \geq 8$, that is equivalently that $|N(u) \setminus N(u')| \leq 2$ and $|N(u') \setminus N(u)| \leq 2$. Note that as $\deg(s) \leq 7$ the set $(N(u) \cap N(u')) \setminus N(s)$ is non-empty, and denote x one of its vertices. By Lemma 35, this vertex x as degree at least 12. As it has exactly 6 neighbors in $N(u)$, at most 2 neighbors in $N(u') \setminus N(u)$, and as it is adjacent to both u and u', x has at least two neighbors in $V(G) \setminus (N[u] \cup N[u'])$. Thus there exists a component $C \neq \{s\}$ in $G \setminus (N[u] \cup N[u'])$. As $C \neq \{s\}$ and $N(C) \subseteq N(u) \cup N(u')$, Lemma 24 implies that $|N(C)| \geq 8$. Therefore, as $|N(u') \setminus N(u)| \leq 2$, $|N(C) \cap N(u)| \geq 6$ and there exist a path P with inner vertices in C and with non-adjacent ends in $N(u)$ (by Claim 34). Let us denote x and y the ends of P. As $|N(u) \cap N(u')| \geq 8$ and by Claim 34, there exists a vertex $z \in N(u) \cap N(u')$ such that $z \neq x$ or y, and such that contracting the edge zu' creates at least

two edges in $N(u)$. As these three added edges (xy and the edges adjacent to z) do not intersect, Claim 33 implies that there is a K_8-minor, a contradiction. $\square$

As by Lemma 24 there is no (≤ 5)-separator (A, B) with $u \in A \setminus B$ and $u' \in B \setminus A$, Menger's Theorem implies the existence of 6 vertex disjoint paths $P_1 \dots P_6$ between u and u'. By minimizing the total length of these paths we can assume that each vertex in $N(u) \cap N(u')$ corresponds to one of these paths, and that any of these paths intersect $N(u)$ (resp. $N(u')$) in only one vertex. Contracting the inner edges (those non-incident to u or u') of these paths, and considering the graph induced by $N[u] \cup N[u']$ one obtains a graph H such that:

- u and u' are nonadjacent and $|N_H(u) \cap N_H(u')| = 6$ or 7.
- $\deg_H(u) = 10$, and $H[N(u)]$ contains $\overline{P_{10}}$ as a subgraph.
- $\deg_H(u') = 10$, and $H[N(u')]$ contains $\overline{P_{10}}$ as a subgraph.

If the graph induced by $N_H(u) \cap N_H(u')$ is isomorphic to $K_{2,2,2}$, then one can assume without loss of generality that $N(u) = \{u_0, \dots, u_9\}$ and that $N(u') = \{u_0, u'_1, u_2, u_3, u'_4, u_5, u_6, u'_7, u'_8, u_9\}$, where the indices correspond to Figure 2. Now observe that contracting the edge u_0u', the path (u_6, u'_7, u'_8), and the path (u_2, u'_4, u'_1), respectively create the edges u_0u_5, u_6u_9, and u_2u_3. This implies by Claim 33 that $N[u]$ contains a K_8-minor, a contradiction. We can thus assume by Claim 34 that the complement of $N_H(u) \cap N_H(u')$ contains a path (a, b, c, d). As $\overline{P_{10}}$ is 6-connected, the graph induced by $\{a, b\} \cup (N_H(u') \setminus N(u))$ is connected, and thus contains a path from a to b. By Claim 33, this path with the path (c, u', d), imply that H (which is a minor of G) contains a K_8-minor, a contradiction. Thus there is no counter-example G, and this concludes the proof of the theorem. $\square$

The proof Theorem 7 is very similar. To do this one can prove the following variant of Theorem 23.

Theorem 37 *Consider a connected K_8-minor free graph G with a vertex s of degree at most 7, such that $N[s] \subsetneq V(G)$ and such that $\min_{v \in V(G) \setminus N[s]} \geq 9$. Then G has an edge $e \in E(G) \setminus E(G[N[s]])$ that belongs to at most 5 triangles.*

The proof of this theorem is as the proof of Theorem 23, except that one does not need to consider the case where some vertex u is such that $N(u)$ induces a $K_{2,2,2,2}$.

5 Global density of triangles

In this section, we investigate the relation between the number of triangles and the number of edge of a graph. Denotes by $\rho = \frac{t}{m}$ the ratio between the number of triangles t and the number of edges m of a graph G. For each k, what is the minimum number $f(k)$ such that for all graph G with $\rho \geq f(k)$, G contains a K_k minor ?

It is easy to notice that 2-trees on $n \geq 2$ vertices have exactly $1+2(n-2)$ edges and $n-2$ triangles. Furthermore, for $k \geq 3$ one can notice that k-trees on $n \geq k$ vertices have exactly $\frac{k(k-1)}{2} + k(n-k)$ edges and $\frac{k(k-1)(k-2)}{6} + (n-k)\frac{k(k-1)}{2}$ triangles. Thus any k-tree, for $k \geq 2$, verifies

$$t = \frac{k-1}{2}m - \frac{1}{2}\binom{k+1}{3}.$$

Since k-trees are K_{k+2}-minor free, for all $k \geq 4$ there exists K_k-minor free graphs with $\frac{k-3}{2}m - \frac{1}{2}\binom{k-1}{3}$ triangles.

We deduce that for all $k \geq 4$, $f(k) \geq \frac{k-3}{2}$. Indeed for every $\epsilon > 0$, there exists a number m and a K_k-minor free graph with m edges such that $\frac{k-3}{2} - \epsilon \leq \rho < \frac{k-3}{2}$. In fact, for $4 \leq k \leq 7$, the following theorem proves that this lower bound is best possible, so we have $f(k) = \frac{k-3}{2}$.

Theorem 38 *For $4 \leq k \leq 7$ (resp. $k = 8$), every graph with $m \geq 1$ edges and $t \geq m(k-3)/2$ triangles has a K_k-minor (resp. a K_8- or a $K_{2,2,2,2,2}$-minor).*

Proof. Consider by contradiction, a non-trivial K_k-minor free (resp. K_8- and $K_{2,2,2,2,2}$-minor free) graph G with $t \geq m(k-3)/2$ triangles. Among the possible graphs G, consider one that minimizes m (given that $m \geq 1$).

Given any edge $uv \in E(G)$ let $H_{uv} = G[N(u) \cap N(v)]$ and denote n' and m' its number of vertices and edges respectively. Contracting uv yields a proper minor of G, with exactly $1 + n'$ edges less, and with at most $n' + m'$ triangles less. Thus by minimality of G, for every edge uv

$$n' + m' > \frac{k-3}{2}(1 + n')$$

which implies that

$$m' > \frac{k-3}{2} + \frac{k-5}{2}n'.$$

On the other hand we have that $\frac{n'(n'-1)}{2} \geq m'$, and this implies that n' should verify $(n'+1)(n'+3-k) > 0$, that is that $n' \geq k-2$. In other words, every edge uv of G belongs to at least $k-2$ triangles. By Theorems 4, (resp. Theorem 8),

this contradicts the K_k-minor freeness (resp. K_8- and $K_{2,2,2,2,2}$-minor freeness) of G. $\square$

6 Application to stress freeness of graphs

The motivation of this application is a problem that arises from the study of tension and compression forces applied on frameworks in the Euclidian space $\mathbb{R}^d$. A d-framework is a graph $G = (V, E)$ and an embedding ρ of G in $\mathbb{R}^d$. The reader should think of a framework as an actual physical system where edges are either straight bars or cables and vertices are articulated joints. A *stress* on a framework (G, ρ) is a function $\omega : E(G) \to \mathbb{R}$ such that $\forall v \in V$,

$$\sum_{\{u,v\}\in E} \omega(\{u, v\})(\rho(v) - \rho(u)) = 0.$$

Stress corresponds to some notion of equilibrium for the associated physical system. Each vertex is affected by tension and compression forces created by the bars and cables. $\omega(\{u, v\})$ can be thought of as the magnitude of such force per unit length, with $\omega(\{u, v\}) < 0$ for a cable tension and $\omega(\{u, v\}) > 0$ for a bar compression. A stress is a state of the system where these forces cancel each other at every vertex. We can see that every framework admits a *trivial* stress where ω is identically zero. A d-framework admitting only the trivial stress is called *d-stress free*.

To make this notion independent of the embedding of G, the following was introduced. A graph G is *generically d-stress free* if the set of all d-stress free embeddings of G in $\mathbb{R}^d$ is open and dense in the set of all its embeddings (i.e. every stressed embedding of G is arbitrary close to a stress free embedding).

This notion has been first used on graphs coming from 1-skeletons of 3-dimensional polytopes [2,14,4,22], which are planar by Steiniz's theorem. Gluck generalized the results on 3-dimensional polytopes to the whole class of planar graphs.

Theorem 39 (Gluck, 1975, [8]) *Planar graphs are generically 3-stress free.*

Nevo proved that we can generalize Theorem 39 for K_5-minor free graphs, and extended the result as follows.

Theorem 40 (Nevo, 2007, [15]) *For $2 \leq r \leq 6$, every K_r-minor free graph is generically $(r-2)$-stress free.*

He conjectured this to hold also for $r = 7$ and noticed that the graph $K_{2,2,2,2,2}$

is an obstruction for the case $r = 8$. Indeed, $K_{2,2,2,2,2}$ is K_8-minor free and has too many edges to be generically 6-stress free (a generically ℓ-stress free graph has at most $\ell n - \binom{\ell+1}{2}$ edges [15]). We answer positively to Nevo's conjecture and we give a variant for the generically 6-stress freeness.

Theorem 41 *Every K_7-minor free graph (resp. K_8- and $K_{2,2,2,2,2}$-minor free graph) is generically 5-stress free (resp. 6-stress free).*

The following result of Whiteley [23] is used to derive Theorem 41.

Theorem 42 (Whiteley, 1989, [23]) *Let G' be obtained from a graph G by contracting an edge $\{u, v\}$. If u, v have at most $d-1$ common neighbors and G' is generically d-stress free, then G is generically d-stress free.*

Now, we prove Theorem 41.

Proof. Assume that G is a K_7-minor free graph (resp. a K_8- and $K_{2,2,2,2,2}$-minor free graph). Without loss of generality, we can also assume that G is connected. Now, contract edges belonging to at most 4 (resp. 5) triangles as long as it is possible and we denotes by G' the graph obtained. Note that by construction, every edge of G' belongs to 5 (resp. 6) triangles. Note also that G' is a minor of G, and is thus K_7-minor free (resp. K_8- and $K_{2,2,2,2,2}$-minor free). Theorem 4 (resp. Theorem 8) thus implies that G' is the trivial graph without any edge and with one vertex. This graph is trivially generically 5-stress free (resp. 6-stress free), and so by Theorem 42, G also is generically 5-stress free (resp. 6-stress free). □

We denote by $\mu(G)$ the Colin de Verdière parameter of a graph G. A result of Colin de Verdière [5] is that a graph G is planar if and only if $\mu(G) \leq 3$. Lovász and Schrijver [12] proved that G is linklessy embeddable if and only if $\mu(G) \leq 4$. Nevo conjectured that the following holds.

Conjecture 43 (Nevo, 2007, [15]) *Let G be a graph and let k be a positive integer. If $\mu(G) \leq k$ then G is generically k-stress free.*

This conjecture holds for the cases $k = 5$ and $k = 6$ as a consequence of Theorem 41.

Corollary 44 *If G is a graph such that $\mu(G) \leq 5$ (resp. $\mu(G) \leq 6$) then G is generically 5-stress free (resp. 6-stress free).*

Proof. Note that $\mu(K_r) = r - 1$ and that if the complement of an n-vertex graph G is a linear forest, then $\mu(G) \geq n-3$ [11]. So we have that $\mu(K_7) = 6$, $\mu(K_8) = 7$, and $\mu(K_{2,2,2,2,2}) \geq 7$.

As the parameter μ is minor-monotone [5], the graph K_7 (resp. K_8 and

$K_{2,2,2,2,2}$) is an excluded minor for the class of graphs defined by $\mu(G) \leq 5$ (resp. $\mu(G) \leq 6$). Hence by Theorem 41, these graphs are generically 5-stress free (resp. 6-stress free). □

7 Application to double-critical k-chromatic graphs

A connected k-chromatic graph is said to be double-critical is for all edge uv of G, $\chi(G \setminus \{u, v\}) = \chi(G) - 2$. It is clear that the clique K_k is such a graph. The following conjecture, known has the Double-Critical Graph Conjecture, due to Erdős and Lovász states that they are the only ones.

Conjecture 45 (Erdős and Lovász, 1968, [6]) *If G is a double-critical k-chromatic graph, then G is isomorphic to K_k.*

This conjecture has been proved for $k \leq 5$ but remains open for $k \geq 6$. Kawarabayashi, Pedersen and Toft have formulated a relaxed version of both Conjecture 45 and the Hadwiger's conjecture, called the Double-Critical Hadwiger Conjecture.

Conjecture 46 (Kawarabayashi, Pedersen, and Toft, 2010, [10]) *If G is a double-critical k-chromatic graph, then G contains a K_k-minor.*

The same authors proved this conjecture for $k \leq 7$ [10], but the case $k = 8$ is left as an open problem. Pedersen proved that every 8-chromatic double-critical contains K_8^- as a minor [17]. Below we prove that the conjecture also holds for $k = 8$.

The following proposition lists some interesting properties about k-chromatic double-critical graphs :

Proposition 47 (Kawarabayashi, Pedersen, and Toft, 2010, [10]) *Let $G \neq K_k$ be a double-critical k-chromatic graph, then*

- *The graph G does not contains K_{k-1} as a subgraph,*
- *The graph G has minimum degree at least $k + 1$,*
- *For all edges $uv \in E(G)$ and all $(k - 2)$-coloring of $G - u - v$, the set of common neighbors of u and v in G contains vertices from every color class.*

In particular, the last item implies that every edge belongs to at least $k - 2$ triangles.

Theorem 48 *Every double-critical k-chromatic graph, for $k \leq 8$, contains K_k as a minor.*

Proof. Consider for contradiction a K_k-minor free graph G that is double-critical k-chromatic. By the second item of Proposition 47, $\delta(G) \geq k+1$. By Theorem 4 and Theorem 7, this graph has an edge that belongs to at most $k-3$ triangles. This contradicts the last item of Proposition 47. $\square$

Let us now give an alternative proof of the case $k = 8$ that does not need Theorem 7, but uses Theorem 8 instead. This might be usefull to prove the next case of Conjecture 46.

Consider for contradiction a K_8-minor free graph G that is double-critical 8-chromatic. By Theorem 8 this graph has an edge that belongs to at most 5 triangles or contains $K_{2,2,2,2,2}$ as an induced subgraph. By Proposition 47 every edge of G belongs to at least 6 triangles, thus G contains $K_{2,2,2,2,2}$ as an induced subgraph. Let us denote $K \subseteq V(G)$ the vertex set of a copy of $K_{2,2,2,2,2}$ in G. As $K_{2,2,2,2,2}$ is maximal K_8-minor free, any connected component C of $G \setminus K$ is such that $N(C) \subset K$ induces a clique. As G is double-critical 8-chromatic, there exists a 6-coloring of $G[N[C]]$, and a 6-coloring of $G \setminus C$. As these two graphs intersect on a clique one can combine their colorings and thus obtain a 6-coloring of G, a contradiction.

8 Application for coloration of K_d-minor free graphs

Hadwiger's conjecture says that every t-chromatic graph G (i.e. $\chi(G) = t$) contains K_t has a minor. This conjecture has been proved for $t \leq 6$, where the case $t = 5$ is equivalent to the Four Color Theorem by Wagner's structure theorem of K_5-minor free graph, and the case $t = 6$ has been proved by Robertson, Seymour and Thomas [19]. The conjecture remains open for $t \geq 7$. For $t = 7$ (resp. $t = 8$) the conjecture asks K_7-minor free graphs (resp. K_8-minor free graphs) to be 6-colorable (resp. 7-colorable). Using Claim 49 and the 9-degeneracy (resp. 11-degeneracy) of these graphs, one can prove that they are 9-colorable (resp. 11-colorable). We improve these bounds by one.

A graph G is said to be *t-minor-critical* if $\chi(G) = t$ and $\chi(H) < t$ whenever H is a strict minor of G. Hadwiger's conjecture can thus be reformulated as follows : Every t-minor-critical graph contains K_t has a minor. In the following $\alpha(S)$ means $\alpha(G[S])$,the independence number of $G[S]$. The following is a folklore claim, here for completeness.

Claim 49 *Given a k-minor critical graph G, for every vertex $v \in V(G)$ we have that* $\deg(v) + 2 - \alpha(N(v)) \geq k$.

Proof. Given a vertex v and a stable set S of $N(v)$, consider the graph G' obtained from G by contracting the edges between v and S. Since G' is a

strict minor of G it is $(k-1)$-colorable. Given such coloring of G', one can $(k-1)$-color $G\setminus\{v\}$ in such a way that all the vertices of S have the same color assigned. In this coloring at most $\deg(v)+1-|S|$ colors are used in $N(v)$, thus $\deg(v)+2-|S|$ colors are sufficient to color G, and thus $\deg(v)+2-\alpha(N(v)) \geq k$. □

A *split graph* is a graph which vertices can be partionned into one set inducing a clique, and one set inducing an independent set. These graphs are the graphs that do not contain C_4, C_5 or $2K_2$ as induced subgraphs [7].

Claim 50 *Given a k-minor critical graph G, every separator (A, B) of G is such that $G[A \cap B]$ is not a split graph (i.e $G[A \cap B]$ contains C_4, C_5 or $2K_2$ as an induced subgraph).*

Proof. Assume by contradiction that there exists such separator (A', B'). This implies the existence of a separator (A, B) such that $S = A \cap B \subseteq A' \cap B'$, and such that each $G[A \setminus S]$ and $G[B \setminus S]$ have a connected component, C_A and C_B such that $N(C_A) = N(C_B) = S$. Note that $G[S]$ is a split graph and let I be one of its maximum independent sets and let $K = S \setminus I$ be a clique. Let G_A and G_B be the graphs respectively obtained from $G[A]$ and $G[B]$ by identifying the vertices of I into a single vertex i. By maximality of I, in both graphs the vertex set $K \cup \{i\}$ induces a clique. Furthermore, these graphs are strict minors of G as the identification of the vertices in I can be done by contracting edges incident to C_B or C_A respectively. Thus, these graphs are $(k-1)$-colorable and these colorings imply the existence of compatible $(k-1)$-colorings of $G[A]$ and $G[B]$, since in both colorings the vertices of I use the same color, and each vertex of K uses a distinct color. This yields in a $(k-1)$-coloring of G, a contradiction. □

Theorem 51 *K_7-minor free graphs are 8-colorable. K_8-minor free graphs are 10-colorable.*

Proof. Consider by contradiction that there is a K_7-minor free graph G non-8-colorable (resp. a K_8-minor free graph G non-10-colorable). This graph is chosen such that $|E(G)|$ is minimal, this graph is thus 9-minor-critical (resp. 11-minor-critical).

For any vertex v, since $\alpha(N(v))$ is at least 1, Claim 49 implies that $\deg(v) > 7$ (resp. $\deg(v) > 9$). If $\deg(v) = 8$ (resp. $\deg(v) = 10$), since G is K_7-minor free (resp. K_8-minor free), we have $\alpha(N(v)) \geq 2$, contradicting Claim 49. Finally if $\deg(v) = 9$ (resp. $\deg(v) = 11$), Claim 49 implies that $3 > \alpha(N(v))$, and since $N(v)$ cannot be a clique, $\alpha(N(v)) = 2$. Thus with Mader's theorem we have that $\delta(G) = 9$ (resp. $\delta(G) = 11$), and that for every vertex v of degree 9 (resp. of degree 11), $\alpha(N(v)) = 2$. By Theorem 4 (resp. Theorem 6), we consider a

vertex u of degree 9 (resp. 11) such that there is an edge uv which belongs to at most 4 (resp. 5) triangles. Let $H = G[N(u)]$, and recall that $\alpha(H) = 2$.

Claim 52 *The graph $H = G[N(u)]$ does not contain a K_5 (resp. a K_6).*

Proof. Assume by contradiction that H contains a K_t with vertices $x_1, \dots, x_t$, for $t = 5$ (resp. for $t = 6$). Assume first that the graph induced by $Y = N(u) \setminus \{x_1, \dots, x_t\}$ is connected. Since $\delta(G) \geq 9$ every vertex x_i has a neighbor in Y or a neighbor w_i in $G \setminus N[u]$. In the latter case, denote C_i the connected component of w_i in $G \setminus N[u]$. Since by Claim 50 (for the partition $(N[C_i], V(G) \setminus C_i)$) $N(C_i)$ intersects Y, one can contract $Y \cup (V(G) \setminus N[u])$ into a single vertex and form a K_{t+2} together with vertices $u, x_1, \dots, x_t$, a contradiction.

Assume now that the graph induced by Y is not connected and let $y_1, y_2 \in Y$ be non-adjacent vertices. Since G is $(2t-1)$-minor critical, consider a $(2t-2)$-coloring of the graph G' obtained from G by contracting uy_1 and uy_2. This coloring implies the existence of a $(2t-2)$-coloring c of $G \setminus u$ such that $c(y_1) = c(y_2)$. As this coloring does not extends to G, the $2t-1$ vertices in $N(u)$ use all the $(2t-2)$ colors. This implies that the colors used for the x_i are used only once in $N(u)$, and that there exists a vertex $z \in Y$ which color is used only once in $N(u)$. Assume $c(x_i) = i$ and $c(z) = 7$. Given two colors a, b and a vertex v colored a, the (a, b)*-component of* v is the the connected component of v in the graph induced by a- or b-colored vertices. For any $1 \leq i \leq t$, suppose we switch colors in the $(i, 7)$-component of z. As this cannot lead to a coloring which does not use all the colors in $N(u)$, there exists a $(7, i)$-bicolored path from z to x_i. This is impossible as contracting these paths on z would lead to a K_{t+2} (with vertex set $\{u, z, x_1, \dots, x_t\}$). This concludes the proof of the claim. □

Let v be a vertex of H with minimum degree in H. By the choice of u and Theorem 4 (resp. Theorem 6), $\deg_H(v) \leq 4$ (resp. $\deg_H(v) \leq 5$).

Claim 53 *$\delta(H) = \deg_H(v) = 4$ (resp. $\delta(H) = \deg_H(v) = 5$).*

Proof. Since $\alpha(H) = 2$, the non-neighbors of v in H form a clique. Furthermore since H does not contain a K_5 (resp. a K_6) we have that $9 - 1 - \deg_H(v) < 5$ (resp. that $11 - 1 - \deg_H(v) < 6$), and hence $\deg_H(v) = 4$ (resp. $\deg_H(v) = 5$). □

Let $y_1, \dots, y_t$ with $t = 4$ (resp. $t = 5$) be the neighbors of v in H, and let K be the t-clique formed by its non-neigbors. By Claim 52 we can assume that y_1 and y_2 are non-adjacent. Note that since $\alpha(G[N(u)]) = 2$ every vertex of K is adjacent to y_1 or y_2. Since G is $(2t+1)$-minor critical, consider a $2t$-coloring of the graph G' obtained from G by contracting uy_1 and uy_2. This coloring

implies the existence of a $2t$-coloring c of $G \setminus u$ such that $c(y_1) = c(y_2)$. As this coloring does not extends to G, the $2t+1$ vertices in $N(u)$ use all the $2t$ colors. In particular, the colors used by K (say $1, \ldots t$) and y_3 (say 6) are thus used only once in $N(u)$. For any $1 \leq i \leq t$, suppose we switch colors in the $(i, 6)$-component of y_3. As this cannot lead to a coloring which does not use all the colors in $N(u)$, there exists a $(i, 6)$-bicolored path from y_3 to the i-colored vertex of K. This is impossible as contracting these paths on y_3, and contracting the edges vy_1 and vy_2 on v would lead to a K_{t+2} with vertex set $\{u, v, y_3\} \cup K$. This concludes the proof of the theorem. □

9 Conclusion

Theorem 38 gives a sufficient condition for a graph to have a K_k-minor. We wonder whether this condition is stronger than Mader's Theorem : Is there a graph G with a K_k-minor, for $4 \leq k \leq 7$, that has $m \leq (k-2)n - \binom{k-1}{2}$ edges and $t \geq m(k-3)/2$ triangles ?

We believe that our work can be extended to the next case. Song and Thomas [20] proved a Mader-like theorem, similar to Theorem 5 in the case of K_9-minor free graphs.

Theorem 54 (Song and Thomas, 2006, [20]) *Every graph on $n \geq 9$ vertices and at least $7n - 27$ edges either has a K_9-minor or is a $(K_{1,2,2,2,2,2}, 6)$-cockade or is isomorphic to $K_{2,2,2,3,3}$.*

Note that $K_{2,2,2,3,3}$ has edges that belong to exactly 6 triangles and contains $K_{2,2,2,2,2,1}$ as a minor. We conjecture that we can extend our main theorem as follows.

Conjecture 55 *Let G a graph such that every edge belongs to at least 7 triangles then either G has a K_9-minor or contains $K_{1,2,2,2,2,2}$ as an induced subgraph.*

Proving this conjecture would have several consequences. This would extend Theorem 38 as follows : Every graph G with $m \geq 1$ edges and $t \geq 3m$ triangles has a K_9 or $K_{1,2,2,2,2,2}$-minor. It would also imply Conjecture 43 for the case $k = 7$, i.e. $\mu(G) \leq 7$ implies that G is generically 7-stress free. Finally, it would imply Conjecture 46 for $k = 9$, i.e. double-critical 9-chromatic graphs have a K_9-minor. We also conjecture that the following holds. In particular, it would imply that K_9-minor free graphs are 12-colorable (using the same arguments as in Section 8).

Conjecture 56 *Any K_9-minor free graph G with $\delta(G) = 13$ has an edge uv*

such that u has degree 13 *and uv belongs to at most* 6 *triangles.*

We also believe that these structural properties on graph with edges belonging to many triangles can actually be extended to matroids. Graph minors can be studied in the more general context of matroid minors [16]. A triangle is then a circuit of size 3. Contrary to graphs, the case when every element of the matroid belongs to 3 triangles is already intricate. There are three well-known matroids for which each element belongs to 3 triangles : the Fano matroid F_7, the uniform matroid $\mathcal{U}_{2,4}$, and the graphical matroid $\mathcal{M}(K_5)$. We conjecture that the following holds.

Conjecture 57 *Let $\mathcal{M}$ be a matroid where each element is contained in* 3 *triangles, then $\mathcal{M}$ admits $\mathcal{M}(K_5)$, F_7 or $\mathcal{U}_{2,4}$ as a minor.*

References

[1] B. Albar and D. Gonçalves. `http://www.lirmm.fr/~albar/triminors/index.html`, 2012.

[2] A. L. Cauchy. Sur les polygones et les polyhèdres. *Second Memoire, I. Ecole Polytechnique*, 1813.

[3] J. Chen and A. Kanevsky. On assembly of four-connected graphs. *Lecture Notes in Computer Science*, 657:158–169, 1993.

[4] H. Crapo and W. Whiteley. Plane self stresses and projected polyhedra i: The basic pattern. *Structural Topology*, 20:55–78, 1993.

[5] Y. Colin de Verdière. Sur un nouvel invariant des graphes et un critère de planarité. *J. Combin. Theory Ser. B*, 50:11–21, 1990.

[6] P. Erdös. Problem 2. *In Theory of Graphs (Proc. Colloq., Tihany, 1966)*, page 361, 1968.

[7] S. Földes and P. L. Hammer. On split graphs and some related questions. *Problèmes Combinatoires et Théorie Des Graphes, Colloques internationaux C.N.R.S.*, 260:138–140, 1976.

[8] H. Gluck. Almost all simply connected closed surfaces are rigid. *Geometric topology. Lecture Notes in Math.*, 438:225–239, 1975.

[9] L. K. Jørgensen. Contractions to K_8. *J. Graph Theory*, 18:431–448, 1994.

[10] K. Kawarabayashi, A. S. Pedersen, and B. Toft. Double-critical graphs and complete minors. *Electron. J. Combin.*, 17(1):Research Paper 87, 27 pp., 2010.

[11] A. Kotlov, L. Lovász, and S. Vempala. The colin de verdière number and sphere representations of a graph. *Combinatorica*, 17:483–521, 1997.

[12] L. Lovász and A. Schrijver. A borsuk theorem for antipodal links and a spectral characterization of linklessly embeddable graphs. *Proc. Amer. Math. Soc.*, 126:1275–1285, 1998.

[13] W. Mader. Homomorphiesätze für Graphen. *Math. Ann.*, 178:154–168, 1968.

[14] J. C. Maxwell. On reciprocal diagrams and diagrams of forces. *Phil. Msg. Series 27*, 4:250–261, 1864.

[15] E. Nevo. Embeddability and Stresses of graphs. *Combinatorica*, 27:465–472, 2007.

[16] J. G. Oxley. *Matroid Theory.* Oxford University Press, New York, 1992.

[17] A. S. Pedersen. Complete and almost complete minors in double-critical 8-chromatic graphs. *Electron. J. Combin.*, 18(1):Research Paper 80, 17 pp., 2011.

[18] N. Robertson and P. D. Seymour. Graph minors. ix. disjoint crossed paths. *J. Combin. Theory Ser. B*, 49:40–77, 1990.

[19] N. Robertson, P. D. Seymour, and R. Thomas. Hadwiger conjecture for k_6-free graphs. *Combinatorica*, 13:279–361, 1993.

[20] Z. Song and R. Thomas. The extremal function for K_9 minors. *J. Combin. Theory Ser. B*, 96:240–252, 2006.

[21] K. Wagner. Uber eine eigenschaft der ebenen komplexe. *Mathematische Annalen*, 114:570–590, 1937.

[22] W. Whiteley. Infinitesimally rigid polyhedra. i. statics of frameworks. *Trans. Amer. Math. Soc.*, 285:431–465, 1984.

[23] W. Whiteley. Vertex splitting in isostatic frameworks. *Structural Topology*, 16:23–30, 1989.

[24] D. R. Wood and S. Linusson. Thomassen's choosability argument revisited. *SIAM J. Discrete Math.*, 24(4):1632–1637, 2010.